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Supplementary Appendix

Abstract

In this supplementary appendix, we present the proof for the necessity for the axioms for the subjective informational tradeoff (SIT) representation, as well as the proof for Lemma 14 in the main text.

S1 Proof for the Necessity of Axioms 1-8

Suppose $\succsim$ has a SIT representation with parameters (π, u, K, σ), we want to show that $\succsim$ satisfies Axioms 1-8. By assumption, $\succsim$ can be represented by

$$V(\mathbb{F}) = \max_{F\in\mathbb{F}} \left[(1+K)\sum_{s\in S}\max_{f\in F}\sum_{\omega\in\Omega}\pi(\omega)\sigma(s \mid \omega)u(f(\omega))\right] - K\sum_{s\in S}\max_{G\in\mathbb{F}}\max_{g\in G}\sum_{\omega\in\Omega}\pi(\omega)\sigma(s \mid \omega)u(g(\omega))$$

It is without loss to assume that $0 \notin S$. For convenience, let

$$U_0(F) := \sum_{s\in S}\max_{f\in F}\sum_{\omega\in\Omega}\pi(\omega)\sigma(s \mid \omega)u(f(\omega)) \tag{S1}$$

$$U_s(F) := \max_{g\in G}\sum_{\omega\in\Omega}\pi(\omega)\sigma(s \mid \omega)u(g(\omega)) \text{ for each } s \in S \tag{S2}$$

It can be easily verified that U_0 and $(U_s)_{s\in S}$ are continuous linear functions from $\mathcal{M}$ to $\mathbb{R}$, where $\mathcal{M}$ is the set of all menus of acts equppied with the Hausdorff metric. Therefore, the binary relation $\succsim$ can be represented by

$$V(\mathbb{F}) = \max_{F\in\mathbb{F}}(1+K)U_0(F) - \sum_{s\in S}\max_{F\in\mathbb{F}} K\cdot U_s(F). \tag{S3}$$

This is a finite DLR type representation over a general convex space.

Lemma S1: *If $\succsim$ has a SIT representation, then $\succsim$ satisfies Axioms 1-3.*

Proof of Lemma 1. By the arguments above, equation (S3) implies that $\succsim$ has a finite DLR type representation over a general domain as characterized in Kopylov (2009). Therefore, $\succsim$ satisfies Axioms 1 and 2 by applying Theorem 2.1 of Kopylov (2009).

Axiom 3, our independence axiom, is slightly stronger than the independence axiom posited in Kopylov (2009). To be precise, we directly verify that $\succsim$ satisfies Axiom 3. Let $\mathbb{F}, \mathbb{G}, \mathbb{H}$ be three directions and $\alpha \in (0, 1)$.

$$\begin{aligned}
\mathbb{F} \succsim \mathbb{G} &\iff V(\mathbb{F}) \geq V(\mathbb{G}) \\
&\iff \alpha V(\mathbb{F}) + (1-\alpha)V(\mathbb{H}) \geq \alpha V(\mathbb{F}) + (1-\alpha)V(\mathbb{H}) \\
&\iff V\big(\alpha\mathbb{F} + (1-\alpha)\mathbb{H}\big) \geq V\big(\alpha\mathbb{G} + (1-\alpha)\mathbb{H}\big) \\
&\iff \alpha\mathbb{F} + (1-\alpha)\mathbb{H} \succsim \alpha\mathbb{G} + (1-\alpha)\mathbb{H}
\end{aligned}$$

where the third equivalence follows from the definition of the convex combination of two directions and the fact that $U_0, (U_s)_{s\in S}$ are all linear functions satisfying

$$U_s(\alpha F + (1-\alpha)G) = \alpha U_s(F) + (1-\alpha)U_s(G)$$

for any menus $F, G \in \mathcal{M}$ and any scalar $\alpha \in [0, 1]$. $\square$

Lemma S2: *If $\succsim$ has a SIT representation, then $\succsim$ satisfies Axioms 4.*

Proof of Lemma 2. Recall that S is a finite set representing the possible signals from the anticipated information structure σ. Let $N = |S| + 2$.

For any direction $\mathbb{F}$ such that $|\mathbb{F}| < N$, the direction itself is a critical subset whose cardinality is smaller than N. For any direction $\mathbb{F}$ such that $|\mathbb{F}| \geq N$ (including the countable and uncountable cases), let $F_0 \in \arg\max_{F\in\mathbb{F}} U_0(F)$, and $F_s \in \arg\max_{F\in\mathbb{F}} U_s(F)$ for each $s \in S$. Then $\mathbb{G} := \{F_0\} \cup \{F_s\}_{s\in S}$ is critical for $\mathbb{F}$ and $|\mathbb{G}| \leq |S| + 1 < N$. For the second part of Axiom 4, fix any direction $\mathbb{F}$ and menu $F \in \mathbb{F}$. If $|F| < N$, then F itself is critical for F in $\mathbb{F}$. If $F \notin \arg\max_{G\in\mathbb{F}} U_0(G)$ and $F \notin \arg\max_{G\in\mathbb{F}} U_s(G)$ for any $s \in S$, then this menu does not matter in the first place for the evaluation of $\mathbb{F}$, so the empty set $\emptyset$ is critical for F in $\mathbb{F}$. Lastly, if $|F| \geq N$ and F is a maximizer of some of U_0 and $(U_s)_{s\in S}$, we discuss two different cases.

Case 1: F is a maximizer of U_0. Let f_s be a maximizer of $\sum_{\omega\in\Omega} \pi(\omega)\sigma(s \mid \omega)u\big(f(\omega)\big)$ for each $s \in S$, then $G := \{f_s\}_{s\in S}$ is critical for F in $\mathbb{F}$, and $|G| \leq |S| < N$.

Case 2: F is a maximizer of U_s for some $s \in S$. Let g be a maximizer of $\sum_{\omega\in\Omega} \pi(\omega)\sigma(s \mid \omega)u\big(f(\omega)\big)$, then $G := \{g\}$ is critical for F in $\mathbb{F}$, and $|G| = 1 < N$. $\square$

Lemma S3: *If $\succsim$ has a SIT representation, then $\succsim$ satisfies Axioms 5.*

Proof of Lemma 3. By equation (S3), $V(\{F\}) = U_0(F)$ for any menu $F \in \mathcal{M}$. Suppose $\{F\} \succsim \{G\}$ and $F \in \mathbb{F}$, we want to show that $\mathbb{F} \succsim \mathbb{F} \cup \{G\}$.

$\{F\} \succsim \{G\}$ implies that $U_0(F) \geq U_0(G)$. Thus for any $\mathbb{F}$ that contains F, $\max_{H\in\mathbb{F}}(1+K)U_0(H) = \max_{H\in\mathbb{F}\cup\{G\}}(1+K)U_0(H)$. That is, $V(\mathbb{F})$ and $V(\mathbb{F}\cup\{G\})$ have the same positive term in equation (S3). Moreover, $\max_{H\in\mathbb{F}} U_s(F) \leq \max_{H\in\mathbb{F}\cup\{G\}} U_s(F)$ for any $s \in S$. Therefore, $V(\mathbb{F}) \geq V(\mathbb{F}\cup\{G\})$, indicating $\mathbb{F} \succsim \mathbb{F}\cup\{G\}$. $\square$

LEMMA S4: *If $\succsim$ has a SIT representation, then $\succsim$ satisfies Axioms 6.*

Proof of Lemma 4. We want to show that $\mathbb{F}\cup\{F\cup G\} \succsim \mathbb{F}\cup\{F,G\}$ for any $\mathbb{F}$ and any F, G. This is easier to see using the alternative expression for the IT representation:

$$\begin{aligned} V(\mathbb{F}) = \max_{F\in\mathbb{F}} &\left[(1+K)\sum_{s\in S}\max_{f\in F}\sum_{\omega\in\Omega}\pi(\omega)\sigma(s\mid\omega)u(f(\omega))\right] \\ &- K\sum_{s\in S}\max_{G\in\mathbb{F}}\max_{g\in G}\sum_{\omega\in\Omega}\pi(\omega)\sigma(s\mid\omega)u(g(\omega)) \end{aligned} \tag{S4}$$

Note that the negative term can be equivalently written as

$$K\sum_{s\in S}\max_{g\in M(\mathbb{F})}\sum_{\omega\in\Omega}\pi(\omega)\sigma(s\mid\omega)u(g(\omega))$$

where $M(\mathbb{F}) = \{f \mid f \in F \text{ for some } F \in \mathbb{F}\}$ denotes the set of feasible acts as defined in the main text. It is clear that $M(\mathbb{F}\cup\{F\cup G\}) = M(\mathbb{F}\cup\{F,G\})$. Therefore, $V(\mathbb{F}\cup\{F\cup G\})$ and $V(\mathbb{F}\cup\{F,G\})$ have the same negative term in equation (S4).

On the other hand, $V(\mathbb{F}\cup\{F\cup G\})$ has a weakly larger positive term because

$$\begin{aligned} &\sum_{s\in S}\max_{f\in F\cup G}\sum_{\omega\in\Omega}\pi(\omega)\sigma(s\mid\omega)u(f(\omega)) \\ \geq\ &\max\left\{\sum_{s\in S}\max_{f\in F}\sum_{\omega\in\Omega}\pi(\omega)\sigma(s\mid\omega)u(f(\omega)), \sum_{s\in S}\max_{f\in G}\sum_{\omega\in\Omega}\pi(\omega)\sigma(s\mid\omega)u(f(\omega))\right\} \end{aligned}$$

Therefore, $V(\mathbb{F}\cup\{F\cup G\}) \geq V(\mathbb{F}\cup\{F,G\})$, indicating $\mathbb{F}\cup\{F\cup G\} \succsim \mathbb{F}\cup\{F,G\}$. $\square$

LEMMA S5: *If $\succsim$ has a SIT representation, then $\succsim$ satisfies Axioms 7 and 8.*

Proof of Lemma 5. For any lottery $\ell \in \Delta(X)$ and its corresponding constant act, $V(\{\{\ell\}\}) = \sum_{\omega\in\Omega}\pi(\omega)u(\ell) = u(\ell)$. Thus, Axiom 7 (Nontriviality) must be satisfied because we require u to be nonconstant.

For Axiom 8 (Domination), suppose f dominates g, then $f(\omega) \succsim g(\omega)$ for any $\omega \in \Omega$, which further implies that $u(f(\omega)) \geq u(g(\omega))$ for any $\omega \in \Omega$. Therefore, $V(\{\{f\}\}) \geq V(\{\{g\}\})$, indicating $f \succsim g$. Moreover, g will never be chosen over f after any signal realization s. Therefore, $V(\{\{f,g\}\}) = \sum_{\omega\in\Omega}\pi(\omega)u(f(\omega)) = V(\{\{f\}\})$, indicating

$\{\{f,g\}\} \sim \{\{f\}\}$. Similarly, if $\mathbb{F}$ dominates $\mathbb{G}$, then $\max_{F\in\mathbb{F}} U_0(F) = \max_{F\in\mathbb{F}\cup\mathbb{G}} U_0(F)$ and for any $s \in S$, $\max_{F\in\mathbb{F}} U_s(F) = \max_{F\in\mathbb{F}\cup\mathbb{G}} U_s(F)$. Therefore, $V(\mathbb{F}) = V(\mathbb{F}\cup\mathbb{G})$, indicating $\mathbb{F} \sim \mathbb{F}\cup\mathbb{G}$. □

S2 Proof for Lemma 14 in the Main Text

The first step is to translate the axioms from $\succsim$ to $\succsim_u$.

AXIOM S1—Weak Order: *$\succsim_u$ is complete and transitive.*

AXIOM S2—Continuity: *$\{\mathbb{G}_u : \mathbb{G}_u \succsim_u \mathbb{F}_u\}$, $\{\mathbb{G}_u : \mathbb{F}_u \succsim_u \mathbb{G}_u\}$ are closed for any $\mathbb{F}_u$.*

AXIOM S3—Independence: *For any $\mathbb{F}_u, \mathbb{G}_u, \mathbb{H}_u$ and any $\alpha \in (0,1)$,*

$$\mathbb{F}_u \succsim_u \mathbb{G}_u \iff \alpha\mathbb{F}_u + (1-\alpha)\mathbb{H}_u \succsim_u \alpha\mathbb{G}_u + (1-\alpha)\mathbb{H}_u.$$

AXIOM S4—Finiteness: *There exists a natural number N such that*

4.1. For any $\mathbb{F}_u \in \mathcal{D}_u$, there exists $\mathbb{G}_u$ with $|\mathbb{G}_u| < N$ such that $\mathbb{G}_u$ is critical[1] *for $\mathbb{F}_u$;*

4.2. For every $\mathbb{F}_u \in \mathcal{D}_u$ and every $F_u \in \mathbb{F}_u$, there exists G_u with $|G_u| < N$ such that G_u is critical for F_u in $\mathbb{F}_u$.

AXIOM S5—Ex-Ante Regret: *If $\{F_u\} \succsim_u \{G_u\}$ and $F_u \in \mathbb{F}_u$, then $\mathbb{F}_u \succsim \mathbb{F}_u \cup \{G_u\}$.*

AXIOM S6—Interim Preference for Flexibility: *$\mathbb{F}_u \cup \{F_u \cup G_u\} \succsim_u \mathbb{F}_u \cup \{F_u, G_u\}$ for any $\mathbb{F}_u$ and any F_u, G_u.*

AXIOM S7—Nontriviality: *There exist $\mathbb{F}_u, \mathbb{G}_u \in \mathcal{D}_u$ such that $\mathbb{F}_u \supseteq \mathbb{G}_u$ and $\mathbb{F}_u \succ_u \mathbb{G}_u$.*

AXIOM S8—Domination: *If f_u dominates g_u, then $f_u \succsim_u g_u$ and $\{\{f_u, g_u\} \sim_u \{\{f_u\}\}$; If $\mathbb{F}_u$ dominates $\mathbb{G}_u$, then $\mathbb{F}_u \sim_u \mathbb{F}_u \cup \mathbb{G}_u$.*

PROPOSITION S1: *If a binary relation $\succsim$ over $\mathcal{D}$ satisfies Axioms 1-8, then a binary relation $\succsim_u$ constructed in the main text satisfies Axioms S1-S8.*

Proof. We can check that the operation $u(\cdot)$ that maps a primitive (acts, menus or directions) into its corresponding primitive as in utilities (utility acts, utility menus and utility directions) behave nicely with set operations and respects linearity. Thus, each Axiom from 1-6 and 8 implies their counterpart in Axiom B.1-B.6 and B.8.

[1]Notions of critical and dominations about $\succsim_u$ are defined in the same way as those about $\succsim$.

Note that the statement of Axiom S7 is slightly different from the statement of Axiom 7. We now check that Axioms 1-8 implies Axiom S7.

By Axiom 7, there exists lotteries $\ell, \ell' \in \Delta(X)$ such that $\ell \succ \ell'$. Thus, ℓ dominates ℓ'. By Axiom 8, $\{\{\ell\},\{\ell'\}\} \sim \{\{\ell\}\}$. Then by transitivity, $\{\{\ell\},\{\ell'\}\} \succ \{\{\ell'\}\}$. Let $\mathbb{F} = \{\{\ell\},\{\ell'\}\}$ and $\mathbb{G} = \{\{\ell'\}\}$. Then $\mathbb{F} \supseteq \mathbb{G}$ and $\mathbb{F} \succ \mathbb{G}$.

Then we can apply the definition of $\succsim_u$ to conclude that $u(\mathbb{F}), u(\mathbb{G}) \in \mathcal{D}_u$, and $u(\mathbb{F}) \supseteq u(\mathbb{G})$ with $u(\mathbb{F}) \succ_u u(\mathbb{G})$, Axiom S7 is satisfied. $\square$

The second step is to translate the axioms from $\succsim_u$ to $\succsim_*$.

Axiom S1*—Weak Order: *$\succsim_*$ is complete and transitive.*

Axiom S2*—Continuity: *$\{\mathbb{G}' : \mathbb{G}' \succsim_* \mathbb{F}'\}$, $\{\mathbb{G}' : \mathbb{F}' \succsim_* \mathbb{G}'\}$ are closed for any $\mathbb{F}'$.*

Axiom S3*—Independence: *For any $\mathbb{F}', \mathbb{G}', \mathbb{H}'$ and any $\alpha \in (0,1)$,*

$$\mathbb{F}' \succsim_* \mathbb{G}' \iff \alpha\mathbb{F}' + (1-\alpha)\mathbb{H}' \succsim_* \alpha\mathbb{G}' + (1-\alpha)\mathbb{H}'.$$

Axiom S4*—Finiteness: *There exists a natural number N such that*

4.1. For every $\mathbb{F}' \in \mathcal{D}'$, there exists $\mathbb{G}'$ with $|\mathbb{G}'| < N$ such that $\mathbb{G}'$ is critical for $\mathbb{F}'$;

4.2. For every $\mathbb{F}' \in \mathcal{D}'$ and every $F' \in \mathbb{F}'$, there exists G' with $|G'| < N$ such that G' is critical for F' in $\mathbb{F}'$.

Axiom S5*—Ex-Ante Regret: *If $\{F'\} \succsim_* \{G'\}$ and $F' \in \mathbb{F}'$, then $\mathbb{F}' \succsim_* \mathbb{F}' \cup \{G'\}$.*

Axiom S6*—Interim Preference for Flexibility: *For any $\mathbb{F}' \in \mathcal{D}'$ and any $F', G' \in \mathcal{M}'$, $\mathbb{F}' \cup \{F' \cup G'\} \succsim_* \mathbb{F}' \cup \{F', G'\}$.*

Axiom S7*—Nontriviality: *There exist $\mathbb{F}', \mathbb{G}' \in \mathcal{D}'$ such that $\mathbb{F}' \supseteq \mathbb{G}'$ and $\mathbb{F}' \succ_* \mathbb{G}'$.*

Axiom S8*—Inclusion: *If $G' \subseteq F'$ and $F' \in \mathbb{F}'$, then $\mathbb{F}' \cup \{G'\} \succsim_* \mathbb{F}'$.*

Note that Axiom S8* (Inclusion) seems to have no couterpart in Axioms S1-S8. As we shall see, it can be implied from part 2 of Axiom S8 (Domination).

Proposition S2: *If $\succsim_u$ satisfies Axioms S1-S8, then $\succsim_*$ satisfies Axioms S1*-S8*.*

Proof that Axiom S1 implies Axiom S1.*
Fix any $\mathbb{F}', \mathbb{G}' \in \mathcal{D}'$. Then by the completeness of $\succsim_u$, either $r(\mathbb{F}') \succsim_u r(\mathbb{G}')$ or $r(\mathbb{G}') \succsim_u r(\mathbb{F}')$ or both, which further indicates that either $\mathbb{F}' \succsim_* \mathbb{G}'$ or $\mathbb{G}' \succsim_* \mathbb{F}'$ or both.

For transitivity, suppose $\mathbb{F}' \succsim_* \mathbb{G}'$ and $\mathbb{G}' \succsim_* \mathbb{H}'$, then by definition, $r(\mathbb{F}') \succsim_u r(\mathbb{G}')$ and $r(\mathbb{G}') \succsim_u r(\mathbb{H}')$, which implies $r(\mathbb{F}') \succsim_u r(\mathbb{H}')$ (by the transitivity of $\succsim_u$), which further implies that $\mathbb{F}' \succsim_* \mathbb{H}'$. $\square$

Proof that Axiom S2 implies Axiom S2.*
Fix any $\mathbb{F}' \in \mathcal{D}'$, consider the set $\{\mathbb{G}' \in \mathcal{D}' \mid \mathbb{F}' \succsim_* \mathbb{G}'\}$. By Axiom S2, the set $\{\mathbb{G}_u \in \mathcal{D}_u \mid r(\mathbb{F}') \succsim_u \mathbb{G}\}$ is closed. Since r is a homeomorphism, it suffices to show that

$$\{\mathbb{G}' \in \mathcal{D}' \mid \mathbb{F}' \succsim_* \mathbb{G}'\} = r^{-1}\Big(\{\mathbb{G}_u \in \mathcal{D}_u \mid r(\mathbb{F}') \succsim_u \mathbb{G}_u\}\Big).$$

If $\mathbb{F}' \succsim_* \mathbb{G}'$, then $r(\mathbb{F}') \succsim_u r(\mathbb{G}')$, thus LHS $\subseteq$ RHS. If $r(\mathbb{F}') \succsim_u \mathbb{G}_u$, then $\mathbb{F}' \succsim_* r^{-1}(\mathbb{G}_u)$, thus LHS $\supseteq$ RHS. The arguments for $\{\mathbb{G}' \in \mathcal{D}' \mid \mathbb{G}' \succsim_* \mathbb{F}'\}$ are similar. $\square$

Proof that Axiom S3 implies Axiom S3.* First note that r is linear, that is, for any $\mathbb{F}', \mathbb{G}' \in \mathcal{D}'$ and $\alpha \in [0,1]$,

$$r(\alpha\mathbb{F}' + (1-\alpha)\mathbb{G}') = \alpha r(\mathbb{F}') + (1-\alpha)r(\mathbb{G}').$$

Then notice that r^{-1} is also linear. For any $\mathbb{F}, \mathbb{G} \in \mathcal{D}$ and $\alpha \in [0,1]$,

$$r^{-1}(\alpha\mathbb{F} + (1-\alpha)\mathbb{G}) = \alpha r^{-1}(\mathbb{F}) + (1-\alpha)r^{-1}(\mathbb{G}).$$

Therefore, for any $\mathbb{F}', \mathbb{G}', \mathbb{H}' \in \mathcal{D}'$ and any $\alpha \in (0,1]$,

$$\begin{aligned}
\mathbb{F}' \succsim_* \mathbb{G}' &\iff r(\mathbb{F}') \succsim_u r(\mathbb{G}') \\
&\iff \alpha r(\mathbb{F}') + (1-\alpha)r(\mathbb{H}') \succsim_u \alpha r(\mathbb{G}') + (1-\alpha)r(\mathbb{H}') \\
&\iff r(\alpha\mathbb{F}' + (1-\alpha)\mathbb{H}') \succsim_u r(\alpha\mathbb{G}' + (1-\alpha)\mathbb{H}') \\
&\iff \alpha\mathbb{F}' + (1-\alpha)\mathbb{H}' \succsim_* \alpha\mathbb{G}' + (1-\alpha)\mathbb{H}'
\end{aligned}$$

This is essentially the same proof as the proof of Claim 3 in DLST. $\square$

Proof that Axiom S4 implies Axiom S4.*
Given $\succsim_u$, let N be a natural number satisfying Axiom S4.

Fix any $\mathbb{F}' \in \mathcal{D}'$, by Axiom S4, there exists $\mathbb{G}_u \in \mathcal{D}_u$ such that $\mathbb{G}_u$ is critical for $r(\mathbb{F}')$ with $|\mathbb{G}_u| < N$. Let $\mathbb{G}' = r^{-1}(\mathbb{G}_u)$. Then $|\mathbb{G}'| = |\mathbb{G}_u| < N$, and for any $\mathbb{H}'$ satisfying $\mathbb{G}' \subseteq \mathbb{H}' \subseteq \mathbb{F}'$, we have

$$r(\mathbb{G}') \subseteq r(\mathbb{H}') \subseteq r(\mathbb{F}') \Longrightarrow r(\mathbb{G}') \sim_u r(\mathbb{H}') \sim_u r(\mathbb{F}') \Longrightarrow G' \sim_* \mathbb{H}' \sim_* \mathbb{F}'.$$

Hence $\mathbb{G}'$ is critical for $\mathbb{F}'$ with $|\mathbb{G}'| < N$. Similar arguments for the second half. $\square$

Proof that Axiom S5 implies Axiom S5.*
First notice that for any $F' \in \mathcal{M}'$, $r(\{F'\}) = \{r(F')\}$ by construction. Therefore,

$$\begin{aligned}
\{F'\} \succsim_* \{G'\}, F' \in \mathbb{F}' &\Longrightarrow \{r(F')\} \succsim_u \{r(G')\}, r(F') \in r(\mathbb{F}') \\
&\Longrightarrow r(\mathbb{F}') \succsim_u r(\mathbb{F}') \cup \{r(G')\} \\
&\Longrightarrow r(\mathbb{F}') \succsim_u r(\mathbb{F}' \cup \{G'\}) \Longrightarrow \mathbb{F}' \succsim_* \mathbb{F}' \cup \{G'\}
\end{aligned}$$

This completes the proof. $\square$

Proof that Axiom S6 implies Axiom S6.*
First note that

$$r(\mathbb{F}' \cup \{F' \cup G'\}) = r(\mathbb{F}') \cup \{r(F') \cup r(G')\}$$

and that

$$r(\mathbb{F}' \cup \{F', G'\}) = r(\mathbb{F}') \cup \{r(F'), r(G')\}.$$

Since $r(\mathbb{F}') \in \mathcal{D}_u$, and $r(F'), r(G') \in \mathcal{M}_u$, then we can use Axiom S6 to conclude that

$$r(\mathbb{F}') \cup \{r(F') \cup r(G')\} \succsim_u r(\mathbb{F}') \cup \{r(F'), r(G')\},$$

which further indicates that $\mathbb{F}' \cup \{F' \cup G'\} \succsim_* \mathbb{F}' \cup \{F', G'\}$. □

Proof that Axiom S7 implies Axiom S7.*
By Axiom S7, there exists $\mathbb{F}_u, \mathbb{G}_u \in \mathcal{D}_u$ such that $\mathbb{F}_u \supseteq \mathbb{G}_u$ and $\mathbb{F}_u \succ_u \mathbb{G}_u$. Therefore, $r^{-1}(\mathbb{F}_u), r^{-1}(\mathbb{G}_u) \in \mathcal{D}'$ satisfy $r^{-1}(\mathbb{F}_u) \supseteq r^{-1}(\mathbb{G}_u)$ and $r^{-1}(\mathbb{F}_u) \succ_* r^{-1}(\mathbb{G}_u)$. □

Proof that Axiom S8 implies Axiom S8.*
Suppose $G' \subseteq F'$, then $r(G') \subseteq r(F')$, and by the definition of domination, $r(F')$ dominates $r(G')$. Suppose $F' \in \mathbb{F}'$, then $r(F') \in r(\mathbb{F}')$, and the second part of Axiom S8 implies that $r(\mathbb{F}') \sim_u r(\mathbb{F}') \cup \{r(G')\}$, which further indicates that $\mathbb{F}' \sim_* \mathbb{F}' \cup \{G'\}$. □

The last step is to translate the axioms from $\succsim_*$ to $\succsim_{**}$.

The following simplifying result helps us use the definition of $\succsim_{**}$ more easily.

Lemma S6: *If $\succsim_u$ satisfies Axioms S1-S3, then*

$$\mathbb{F}'' \succsim_{**} \mathbb{G}'' \iff \frac{1}{n^2}\mathbb{F}'' + \left(1 - \frac{1}{n^2}\right)\{F^{n+1}\} \succsim_* \frac{1}{n^2}\mathbb{G}'' + \left(1 - \frac{1}{n^2}\right)\{F^{n+1}\}.$$

Proof of Lemma 6. Since $\succsim_u$ satisfies Axioms S1-S3, $\succsim_*$ satisfies Axioms S1*-S3*. Recall that $F^{n+1} = \left\{\left(\frac{n}{n+1}, \ldots, \frac{n}{n+1}\right)\right\}$ is the singleton menu containing only a "uniform lottery."

"$\Longleftarrow$": Suppose $\frac{1}{n^2}\mathbb{F}'' + \left(1 - \frac{1}{n^2}\right)\{F^{n+1}\} \succsim_* \frac{1}{n^2}\mathbb{G}'' + \left(1 - \frac{1}{n^2}\right)\{F^{n+1}\}$. Since $\{F^{n+1}\} \in \mathcal{D}'$ and $\succsim_*$ satisfies B.3*, for any $\varepsilon \in [0, 1/n^2)$, $\alpha := n^2\varepsilon \in [0, 1)$ and

$$\begin{aligned}
\varepsilon\mathbb{F}'' + (1-\varepsilon)\{F^{n+1}\} &= \alpha\left(\frac{1}{n^2}\mathbb{F}'' + \left(1 - \frac{1}{n^2}\right)\{F^{n+1}\}\right) + (1-\alpha)\{F^{n+1}\} \\
&\succsim_* \alpha\left(\frac{1}{n^2}\mathbb{G}'' + \left(1 - \frac{1}{n^2}\right)\{F^{n+1}\}\right) + (1-\alpha)\{F^{n+1}\} \\
&= \varepsilon\mathbb{G}'' + (1-\varepsilon)\{F^{n+1}\}
\end{aligned}$$

"$\Longrightarrow$": Suppose $\mathbb{F}'' \succsim_{**} \mathbb{G}''$, then we can find a sequence of ε_n converging from below to $1/n^2$, then the result follows from the continuity of $\succsim_*$. □

An immediate but important implication of Lemma 6 is that the constructed binary relation $\succsim_{**}$ is uniquely determined by $\succsim_{*}$.

Moving forward, we will repeat the operation "mix $\mathbb{F}''$ with $\{F^{n+1}\}$ on weight $1/n^2$" several times. For convenience, we define this operation by $t : \mathcal{D}'' \to \mathcal{D}'$ where

$$t(\mathbb{F}'') := \frac{1}{n^2}\mathbb{F}'' + \left(1 - \frac{1}{n^2}\right)\{F^{n+1}\} \tag{S5}$$

PROPOSITION S3: *If $\succsim_{*}$ satisfies Axioms S1*-S8*, then $\succsim_{**}$ satisfies Axioms 1*-8*.*

Proof that Axioms S1-S3* imply Axiom 1*.*

Completeness: Fix any $\mathbb{F}'', \mathbb{G}'' \in \mathcal{D}''$, $t(\mathbb{F}''), t(\mathbb{G}'') \in \mathcal{D}'$. By the completeness of $\succsim_{*}$, $t(\mathbb{F}') \succsim_{*} t(\mathbb{G}')$ or $t(\mathbb{G}') \succsim_{*} t(\mathbb{F}')$ or both. Moreover, Axioms S1*-S3* guarantee that $\mathbb{F}'' \succsim_{**} \mathbb{G}''$ if and only if $t(\mathbb{F}'') \succsim_{*} t(\mathbb{G}'')$. Thus, $\succsim_{**}$ is complete. For transitivity,

$$\begin{aligned}
\mathbb{F}'' \succsim_{**} \mathbb{G}'',\ \mathbb{G}'' \succsim_{**} \mathbb{H}'' &\Longrightarrow t(\mathbb{F}'') \succsim_{*} t(\mathbb{G}''),\ t(\mathbb{G}'') \succsim_{*} t(\mathbb{H}'') \\
&\Longrightarrow t(\mathbb{F}'') \succsim_{*} t(\mathbb{H}'') \\
&\Longrightarrow \mathbb{F}'' \succsim_{**} \mathbb{H}''
\end{aligned}$$

where the second implication follows from $\succsim_{*}$ being transitive. □

Proof that Axioms S1-S3* imply Axiom 2*.*

We consider the lower contour set. Same arguments work for the upper contour set.

Fix any sequence $\{\mathbb{G}''_m\}_{m=1}^{\infty} \subset \{\mathbb{G}'' \mid \mathbb{F}'' \succsim_{**} \mathbb{G}''\}$, that is, $\mathbb{F}'' \succsim_{**} \mathbb{G}''_m$ for all $m \in \mathbb{N}$, which further indicates that $t(\mathbb{F}'') \succsim_{*} t(\mathbb{G}''_m)$ for all $m \in \mathbb{N}$. If $\mathbb{G}''_m \to \mathbb{G}''_*$ as $m \to \infty$ for some $\mathbb{G}''_* \in \mathcal{D}''$, then by construction, $t(\mathbb{G}''_m) \to t(\mathbb{G}''_*)$. Since $\succsim_{*}$ satisfies Continuity (Axiom S2*), this implies that $t(\mathbb{F}'') \succsim_{*} t(\mathbb{G}''_*)$. Thus, $\mathbb{F}'' \succsim_{**} \mathbb{G}''_*$. □

Proof that Axioms S1-S3* imply Axiom 3*.*

First note that the mapping $t : \mathcal{D}'' \to \mathcal{D}'$ is linear, that is, for any $\alpha \in [0, 1]$,

$$\begin{aligned}
t\Big(\alpha\mathbb{F}'' + (1-\alpha)\mathbb{G}''\Big) &= \frac{1}{n^2}\Big(\alpha\mathbb{F}'' + (1-\alpha)\mathbb{G}''\Big) + \left(1 - \frac{1}{n^2}\right)\{F^{n+1}\} \\
&= \frac{1}{n^2}\Big(\alpha\mathbb{F}'' + (1-\alpha)\mathbb{G}''\Big) + \left(1 - \frac{1}{n^2}\right)\Big(\alpha\{F^{n+1}\} + (1-\alpha)\{F^{n+1}\}\Big) \\
&= \alpha t(\mathbb{F}'') + (1-\alpha)t(\mathbb{G}'')
\end{aligned}$$

Note that the second equality only goes through because F^{n+1} is a singleton menu. The decomposition is not generally doable for non-singleton menus.

Using the linearity of t, we prove the following stronger version of Independence. For any $\mathbb{F}'', \mathbb{G}'', \mathbb{H}''$ and any $\alpha \in (0,1]$,

$$\begin{aligned} \mathbb{F}'' \succsim_{**} \mathbb{G}'' &\iff t(\mathbb{F}'') \succsim_{*} t(\mathbb{G}'') \\ &\iff \alpha t(\mathbb{F}'') + (1-\alpha) t(\mathbb{H}'') \succsim_{*} \alpha t(\mathbb{G}'') + (1-\alpha) t(\mathbb{H}'') \\ &\iff t\Big(\alpha \mathbb{F}'' + (1-\alpha)\mathbb{H}''\Big) \succsim_{*} t\Big(\alpha \mathbb{G}'' + (1-\alpha)\mathbb{H}''\Big) \\ &\iff \alpha \mathbb{F}'' + (1-\alpha)\mathbb{H}'' \succsim_{**} \alpha \mathbb{G}'' + (1-\alpha)\mathbb{H}'' \end{aligned}$$

where the first and the last equivalences follow from the previous lemma, the second equivalence follows from $\succsim_*$ satisfying Axiom S3* and the third equivalence follows from the linearity of t we showed above. $\square$

Proof that Axioms S1-S4* imply Axiom 4*.*
First note that the mapping t respects set inclusion, that is, if $\mathbb{G}'' \subseteq \mathbb{F}''$, then

$$t(\mathbb{G}'') = \frac{1}{n^2}\mathbb{G}'' + \left(1 - \frac{1}{n^2}\right)\{F^{n+1}\} \subseteq \frac{1}{n^2}\mathbb{F}'' + \left(1 - \frac{1}{n^2}\right)\{F^{n+1}\} = t(\mathbb{F}'').$$

Moreover, $|t(\mathbb{F}'')| = |\mathbb{F}''|$ for any finite $\mathbb{F}''$.

Now Axiom S4 implies Axiom S4*, which guarantees the existence of a natural number N satisfying the conditions for $\succsim_*$ over $\mathcal{D}'$ corresponding to the two conditions above. We argue the same N works for $\succsim_{**}$.

Fix any $\mathbb{F}'' \in \mathcal{D}''$, $t(\mathbb{F}'') \in \mathcal{D}'$, and thus by Axiom S4*, there exists $\mathbb{G}'$ such that $\mathbb{G}'$ is critical for $t(\mathbb{F}'')$ and $|\mathbb{G}'| < N$. Since $\mathbb{G}' \subseteq t(\mathbb{F}'')$, there exists $\mathbb{G}'' \subseteq \mathbb{F}''$ such that $\mathbb{G}' = t(\mathbb{G}'')$. So $|\mathbb{G}''| = |\mathbb{G}'| < N$, and we argue that $\mathbb{G}''$ is critical for $\mathbb{F}''$. Fix any $\mathbb{H}''$ such that $\mathbb{G}'' \subseteq \mathbb{H}'' \subseteq \mathbb{F}''$, then $\mathbb{G}' = t(\mathbb{G}'') \subseteq t(\mathbb{H}'') \subseteq t(\mathbb{F}'')$. This implies that $G' \sim_* t(\mathbb{H}'') \sim_* t(\mathbb{F}'')$ since $\mathbb{G}'$ is critical for $t(\mathbb{F}'')$, which further implies that $\mathbb{H}'' \sim_{**} \mathbb{F}''$.

Similar arguments work for the second condition. $\square$

Proof that Axioms S1-S3* and S5* imply Axiom 5*.*
First note that the mapping $t : \mathcal{D}'' \to \mathcal{D}'$ respects set unions: For any $\mathbb{F}'', \mathbb{G}'' \in \mathcal{D}''$,

$$\begin{aligned} t\Big(\mathbb{F}'' \cup \mathbb{G}''\Big) &= \frac{1}{n^2}\Big(\mathbb{F}'' \cup \mathbb{G}''\Big) + \left(1 - \frac{1}{n^2}\right)\{F^{n+1}\} \\ &= \left(\frac{1}{n^2}\mathbb{F}'' + \left(1 - \frac{1}{n^2}\right)\{F^{n+1}\}\right) \cup \left(\frac{1}{n^2}\mathbb{G}'' + \left(1 - \frac{1}{n^2}\right)\{F^{n+1}\}\right) \\ &= t(\mathbb{F}'') \cup t(\mathbb{G}'') \end{aligned}$$

Now suppose $\mathbb{F}''$ and $\mathbb{G}''$ satisfy that for any $G'' \in \mathbb{G}''$, there exists $F'' \in \mathbb{F}''$ such that $\{F''\} \succsim_{**} \{G''\}$, we want to show that $\mathbb{F}'' \succsim_{**} \mathbb{F}'' \cup \mathbb{G}''$.

For any $G' \in t(\mathbb{G}'')$, there exists $G'' \in \mathbb{G}''$ such that $\{G'\} = t(\{G''\})$. By assumption, there exists $F'' \in \mathbb{F}''$ such that $\{F''\} \succsim_{**} \{G''\}$. Then $t(\{F''\}) \succsim_{*} t(\{G''\}) = \{G'\}$ and $t(F'') \in t(\mathbb{F}'')$. (This is an abuse of notation, by $t(F'')$ we mean the only menu contained in $t(\{F''\})$.) Then the assumption in Axiom S5* is satisfied and we can conclude that $t(\mathbb{F}'') \succsim_{*} t(\mathbb{F}'') \cup t(\mathbb{G}'') = t(\mathbb{F}'' \cup \mathbb{G}'')$, which further indicates that $\mathbb{F}'' \succsim_{**} \mathbb{F}'' \cup \mathbb{G}''$. $\square$

Proof that Axioms S1-S3* and S6* imply Axiom 6*.*
We first formalize the notation (abuse) appeared above. Let $F'' \in \mathcal{M}''$ be a menu, then

$$t(F'') := \frac{1}{n^2} F'' + \left(1 - \frac{1}{n^2}\right) F^{n+1} \in \mathcal{M}'.$$

And $t(F'' \cup G'') = t(F'') \cup t(G'')$ for all $F'', G'' \in \mathcal{M}''$. Now for any $\mathbb{F}'' \in \mathcal{D}''$ and any $F'', G'' \in \mathcal{M}''$, $t(\mathbb{F}'') \in \mathcal{D}'$, $t(F''), t(G'') \in \mathcal{M}'$. Thus, by Axiom S6*,

$$t(\mathbb{F}'') \cup \{t(F'') \cup t(G'')\} \succsim_{*} t(\mathbb{F}'') \cup \{t(F''), t(G'')\}.$$

But $t\Big(\mathbb{F}'' \cup \{F'' \cup G''\}\Big) = t(\mathbb{F}'') \cup \{t(F'' \cup G'')\} = t(\mathbb{F}'') \cup \{t(F'') \cup t(G'')\}$, and similar for the RHS. Therefore, $\mathbb{F}'' \cup \{F'' \cup G''\} \succsim_{**} \mathbb{F}'' \cup \{F'', G''\}$. $\square$

Proof that Axioms S1-S3* and S7* imply Axiom 7*.*
If $F'' \supseteq G''$, then $t(F'') \supseteq t(G'')$, and by Axiom S7*, $t(\mathbb{F}'') \cup \{t(F''), t(G'')\} \succsim_{*} t(\mathbb{F}'') \cup \{t(F'')\}$. $\square$

References

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